Mathematics, Art and Science of the Pseudosphere

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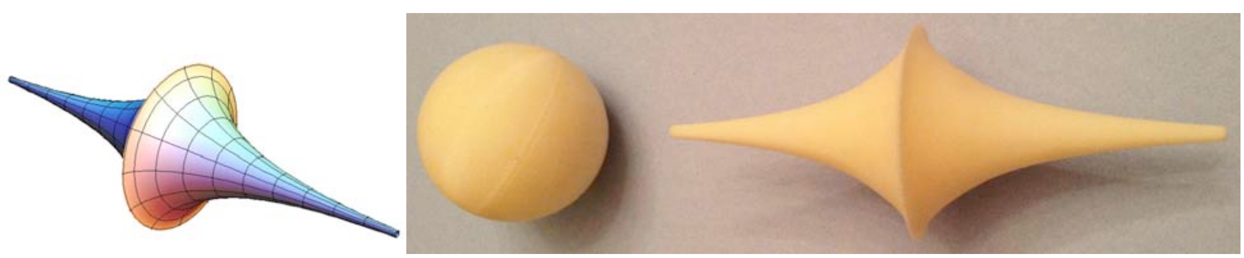
Abstract

The pseudosphere is mainly thought of as an example of a mathematical surface that exemplifies non-Euclidean geometries. It has also been incorporated into a variety of 20th century works of art and has applications in science.

The Pseudosphere: What is it? Who Invented (or Discovered) it? When? Where?

A pseudosphere is a surface with constant negative Gaussian curvature (see Fig. 1a). It was first discussed extensively in two papers published in 1868 by the Italian mathematician Eugenio Beltrami, who gave this surface its name. However, Christiaan Huygens had written about the object and some of its properties as early as 1693. Revolving a tractrix of radius r about its asymptote generates the surface. Although the resulting surface (sometimes called a tractroid) has infinite extension along its central axis, it has finite area $A = 4\pi r^2$ and volume $V = (2/3)\pi r^3$ (see Fig. 1b). It has exactly the same surface area as a sphere of radius r, and half its volume. Beltrami introduced the pseudosphere in his analysis and discussion of the work that had been done on non-Euclidean geometry by Hungarian mathematician Janos Bolyai and Russian mathematician Nicolai Lobachevsky a half century earlier. This more easily graspable example of a non-Euclidean space of constant negative curvature played a major role in subsequent papers by the German polymath Hermann von Helmholtz, the French mathematician Henri Poincare and others that laid to rest the philosophical idea of the primacy (and even absolute necessity) of Euclidean geometry that had persisted for over 2000 years.

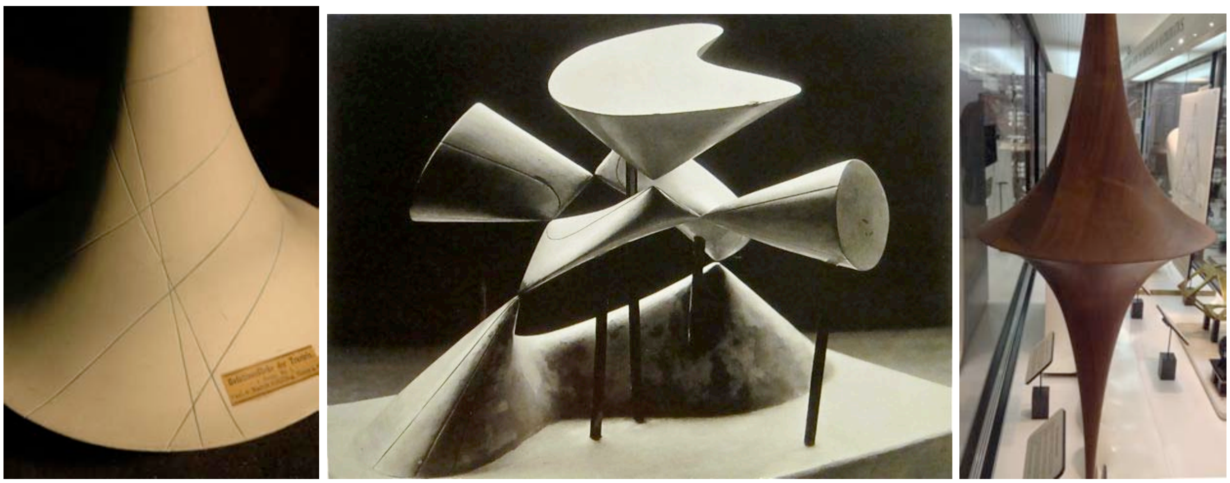


Figures 1 a (l.) & b (r.): *(a) A computer drawing of a truncated pseudospherical surface. (b) A sphere and a pseudosphere, each of radius r = 1.5". These models were made using rapid prototyping methods.*

In the 20th century, as cosmological applications of General Relativity expanded the realm of non-Euclidean geometry to the domains of physics and astronomy, the subject also penetrated other intellectual fields such as psychology (e.g., the idea of visual space being non-Euclidean) and literature (e.g., Rene Daumal's "Mount Analogue"). However, despite the tangibility of the pseudosphere, the usual textbook example of a non-Euclidean space of negative curvature has been the hyperbolic parabaloid.

Physical Embodiments of the Mathematical Pseudosphere

Beginning in the 1870's, Felix Klein and several other (primarily German) mathematicians began to encourage the use of physical realizations of otherwise abstract mathematical functions and surfaces (see Fig. 2 a and Fig. 2 b). Such mathematical models became quite popular as aids in teaching, and could be found in university mathematics departments throughout Europe, the United States and even Japan in the late 19th and early 20th centuries [1]. However, by the 1930's such models were no longer manufactured. And with the rise of the Bourbaki anti-visualization school of mathematics, the models had subsequently been relegated to museums or to dustbins.



Figures 2 a (l.), b (c.) & c (r.): *(a) Late 19th century plaster model of a pseudosphere made by Verlag Martin Schilling; (b) 1936 photograph by Man Ray of a Kummer surface plaster model in the collection of the Institute Henri Poincare; (c) Laminated wood pseudosphere model currently on display in the Boston Museum of Science exhibit entitled "Mathematica" that was created by Charles and Ray Eames.*

Two-dimensional images of mathematical functions re-entered science and mathematics education with the advances in the capabilities of computer graphics in the 1970's. The new millennium has brought with it the ability to fabricate three-dimensional mathematical models far more easily than was possible in the 19th century. We have now made easily reproducible models of the pseudosphere for use in astronomy, physics and mathematics education by employing rapid prototyping techniques (see Fig. 1b).

The Pseudosphere in 20th and 21st Century Art

How non-Euclidean geometries and mathematical models expanded their domain from mathematics to science and then to the visual arts is not totally agreed upon by either art historians or by historians of science and mathematicians [2]. Some authors say that the extension occurred when Max Ernst suggested to Man Ray that he visit the Institut Henri Poincare to see the mathematical models exhibited there. Between 1934 and 1936, he shot a suite of photographs he entitled "Objet Mathematique". Many were intriguingly shot from a low angle giving them an otherworldly surrealist cast. Some of these images (like Fig. 2 b) were published in the art journal "Cahiers d' Art" in 1936 and then went on to reach the wider art world. Man Ray himself later produced a number of paintings incorporating some of the mathematical models he saw. The British sculptor Henry Moore also made a number of sculptures that were based on specific mathematical models he saw in the South Kensington Science Museum in London, though neither artist seems to have explicitly incorporated the pseudosphere into his works of art.

After World War II there was an explosion of art movements including Minimalism. The pseudosphere began to appear in various kinds of art works, including engravings and photographs, though its most visible appearance has been in sculpture. Anish Kapoor has made several versions of the pseudosphere in different scale sizes for placement in either indoor galleries or in outdoor pubic spaces (see Fig 3a). Japanese artist Hiroshi Sugimoto, who was inspired by a set of the German 19^{th} century mathematical models on display at Tokyo University, has emulated Man Ray's photographs with his own photographic study of the models (e.g., Fig. 3 b), and has created his own three dimensional versions (Fig. 3c.)

Figures 3 a (l.), b (c.) and c (r.): *(a) A. Kapoor stainless steel sculpture "Non Object (Spire)" 2008; (b) H. Sugimoto silver gelatin print of a* 19^{th} *century model of a "Surface of Revolution of Constant Negative Curvature", 2004; (c) H. Sugimoto aluminum and glass sculpture "Conceptual Form 009", 2006.*

Perhaps the artist who used pseudospheres most prominently in her art was Ruth Vollmer [3]. This largely self-taught artist had immigrated to the United States prior to World War II. She discovered the 19^{th} century mathematical models on display at Columbia University and made versions of some of them utilizing a variety of materials including wood, aluminum, bronze and acrylic.

Figures 4 a (l.) and b (r.): *(a) Laminated wood sculpture entitled "Pseudosphere" by Ruth Vollmer (b) Metal and wire sculpture "Funicular Polygon of Revolution – Pseudosphere" by Robert Le Ricolais.*

But are they "Art"? The American minimalist artist Sol LeWitt, writing about the sculptures of Ruth Vollmer in the journal "Studio International", had this to say about her work: "These pieces are not sculpture: they are ideas made into solid forms…. The pieces are not about mathematics; they are about art." Whatever these things are, they are intriguing objects that straddle multiple disciplines.

The Pseudosphere Uphill Roller: a New Scientific Demonstration

The "Uphill Roller" is a beautiful (and counterintuitive) physics demonstration. English mathematician William Leybourn first reported it in 1694 (see Fig. 5a). In the original version (see Fig. 5b), a double cone placed on two divergent inclined ramps appears to roll "uphill", apparently violating the laws of physics. Actually, it is an example of an object's center of mass *descending* under the influence of gravity [4]. In the succeeding three centuries, this widely used lecture and museum demonstration has remained essentially the same. However, if one replaces the double cone with a pseudosphere, then something *really* counter-intuitive can occur. Depending on the angles of inclination and divergence of the ramps, the pseudosphere can ascend *or* oscillate back and forth, finally reaching an equilibrium position near the center of the ramp (see Figure 5c). The explanation for this effect is left as an exercise for the reader.

Figures 5 a (l.), b (c.) & c (r.): *(a) The original 1694 description of "A Mechanical Paradox" employing a double cone; (b) 18th century wood and brass version displayed in the Museo Galileo in Florence, Italy; (c) Our plastic rapid prototype pseudosphere in its equilibrium position on a wooden double ramp.*

Summary

The pseudosphere played an important mathematical role in the acceptance of non-euclidean geometry. Beginning in the 20th century, it was also utilized in a variety of works of fine art. We report here for the first time a new "roll" for the pseudosphere in a three-century-old classic physics demonstration.

Acknowledgments

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