Top-ology: A Torque about Tops

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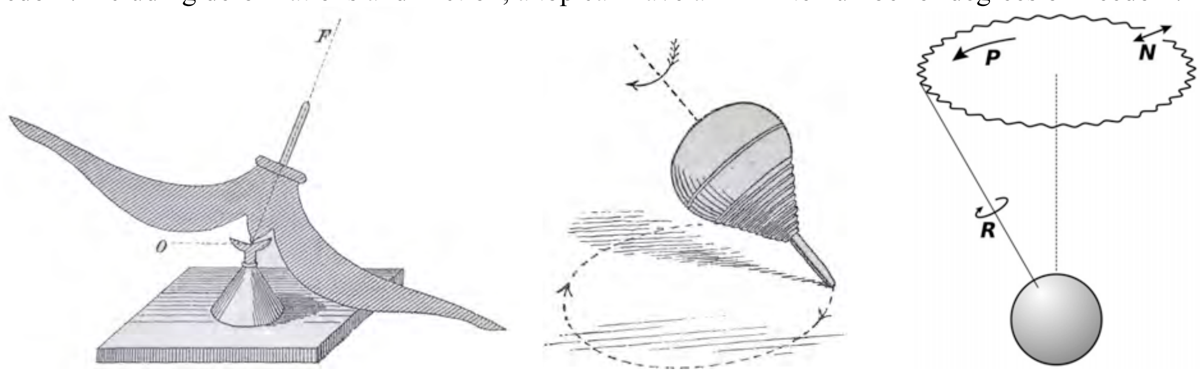
Abstract

Spinning tops have been part of human civilization for at least three millennia. They have intrigued and mystified artists and artisans as well as mathematicians and physicists. Besides rotating about an axis, tops can precess and nutate, as well as shake, rattle and roll. Perhaps this is why children have included tops in so much of their play. And some of the greatest physicists of all times have turned their attention to them, including Isaac Newton, James Clerk Maxwell and Albert Einstein. This paper presents a brief overview of the history of tops, their designs, and their incorporation into paintings, sculpture, literature, film and music. It also looks at some of their applications in fields as diverse as navigation and neuroscience. It concludes with the introduction of a new type of top that combines both spin and wobble into a kind of motion that might appropriately be called "spobbling".

What Is A Top?

In their monumental study "The Theory of the Top" [4], Felix Klein and Arnold Sommerfeld wrote "By a top we mean - reserving a later development of the concept - a rigid body subject to gravity whose mass is symmetrically distributed about an axis of the body, and which, by means of an appropriate device, is fixed in space at one point of the symmetry axis." Fig. 1a is the first figure in Volume 1 of their study, which grew to four volumes, running nearly a thousand pages in length. Their treatise was published over a 14-year period from 1897 to 1910. Klein and Sommerfeld initially restricted their considerations of tops to those with only two or three degrees of freedom that could be analyzed with the mathematical tools then available. A "peg top" (Fig. 1b) that is allowed to move on a horizontal plane has five degrees of freedom. Including deformations and friction, a top can have an infinite number of degrees of freedom.



Figures 1a (l.), 1b (c.) & 1c (r.): *(a) Cross section of a top constrained by a fixed point; (b) a "peg top" moving freely on a horizontal plane; (c) illustration of top spin (R), precession (P) and nutation (N).*

Taking a broader point of view, almost any object can act as a top (except the Higgs boson which has no spin). A top need not be symmetrical, rigid or fixed in space. Electrons, the Earth and the largest spiral galaxies can all be viewed and studied as spinning tops. A spinning ball can be considered to be a top. So can a disk such as an Olympic style discus or a modern Frisbee. Hoops (used in hoop rolling), yo-yos and diabolos also can be considered to be tops. Objects that can exhibit more complex motions (beyond the spin, precession and nutation (Fig. 1c) exhibited by simple rigid symmetrical tops) such as tipping, rolling, rattling and wobbling are discussed at the end of this paper.

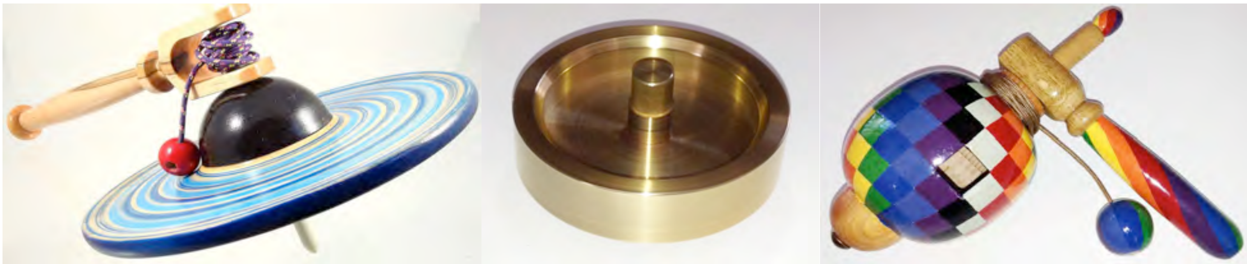
A Brief History of Tops

Spinning tops have appeared in most ancient cultures. One of the earliest surviving objects that definitely can be identified as a top was found in the tomb of the Egyptian pharaoh Tutankhamen (Fig. 2a). Tops come in many sizes and designs. The mechanism that sets them spinning - and keeps them spinning - also varies widely. Fig. 2b shows a Hittite bas-relief found in Turkey showing a "whip top" which is kept spinning by the application of a hand held whip. The image from a Greek vase shown in Fig. 2c shows a person yanking up a yo-yo. Early tops seem to have had two purposes: play (or sport) and gaming (or gambling). When used for play, the goal is often spin duration. This activity continues to the present day in the Malaysian state of Kelantan where large throwing tops 30 - 40 cm across can be made to spin for as long as 1 - 2 hours. Other cultures have used tops in competitions where the goal was to knock the opponent's top out of a prescribed area. When used for gaming, they are often referred to as "teetotums". The Jewish dreidel with four sides is such an ancient top design. So are spinning tops with six or more sides that can be used in place of dice. The book "The Top" by D. W. Gould [3] provides a fine overview of their long history that continues to the present day in some cultures. "The Lost Art of Spinning Tops" by Lourens Bas [2] is a magnificent book, surely the most beautiful ever written on the subject of spinning tops. Its title no doubt refers to the disappearance of play with tops in most western societies, presumably resulting from the triumph of video games over toys that involve any physical activity.

Figures 2a (l.), 2b (c.) & 2c (r.): *(a) Top from the tomb of Tutankhamen ca.1300 BC; (b) Hittite bas-relief ca 800 BC (?); (c) image of a yo-yo on a Greek vase ca. 500 BC.*

A Short Taxonomy of Top Types

There are many kinds of tops. The book by Gould [3] distinguishes: twirlers (finger tops), peg tops, supported tops, whip tops and several others. Some are identified by their means of propulsion: supported tops (Fig. 3a) are spun by first wrapping a string around them and then spinning them up using an external support; finger tops (Fig. 3b) are spun by a snap of the fingers. Others are named for the effect they produce: humming (or whistling) tops are hollow and have a hole in the side of the top (Fig. 3c) and they hum or whistle when spun! Peg tops and whipping tops are first wrapped by a string and are then thrown, the latter kept spinning by whipping them. If one includes objects like yo-yos, hoops, boomerangs and diabolos there are probably dozens of different types of tops discussed in the literature.



Figures 3a (l.), 3b (c.) & 3c (r.): *(a) Black hole top (KB design, made by R. Bischoff, 22 cm dia.); (b) brass finger top (KB design, made by BU shop, 6cm dia.); (c) wood humming top (by E. Horne, 7cm dia.)*

Some Physical Considerations

One of the things that make a top so engaging is its counter-intuitive behavior. It seems to defy gravity by not falling over, and it can precess about its spin axis (and also nod or nutate). Generally, if one pushes or pulls a non-rotating object, it responds by moving in a straight line. However, impressing a force on a spinning object results in a motion perpendicular to the direction of the force. To understand tops, the central physical issue arises from the fact that a vector force **F** (here a bold faced letter indicates a vector) applied from a distance **r** through the center of a spinning top will result in a torque **N** acting on the top**.** If the top has a moment of inertia I and spins with angular frequency **ω**, it will have angular momentum **L** = I**ω**. The torque will make the object precess about its spin axis so that it satisfies the relation **F** x **r** = **N** = d**L**/dt. A detailed analysis of the effect of this torque acting on a top spinning on a fixed point (the kind of top discussed by Klein and Sommerfeld) shows that two kinds of motions can result: precession of the spin axis and a nodding up and down of the of the top (nutation). Though Isaac Newton originally published the laws of mechanics in his "Principia" in 1687, he never achieved the same level of understanding of rotational motion as he did of linear motion. Those insights have required the work of mathematicians, "natural philosophers" and mathematical physicists over the succeeding three centuries.

From Top-ology to Topology

Following Newton's development of his theory of mechanics, the motion of a rigid body became a major theme in classical mechanics. The motions of the planets, their satellites and rings were perhaps the pre-eminent dynamical problems of the 18th century. However, the motion of spinning tops also posed challenges to mathematicians and physicists. The complete solution for such problems requires being able to find the constants of the motion in closed form. In the case of celestial mechanics, no completely integrable solutions exist beyond the case of two bodies orbiting one another. Even this case is not tractable once tidal interactions and friction come into play. For the motion of tops governed by the innocuous looking torque equation in the preceding section, only three very special cases can be fully integrated in closed form to completely specify their motion [1, 4]. Leonard Euler investigated the first case beginning in 1751. This involves the motion of a top moving freely without external torque. The resulting Eulerian free precession is exhibited by the Earth (Chandler wobble) and even by neutron stars. In 1811 Joseph-Louis Lagrange derived the motion of a symmetric top that moves about a fixed point in which the center of gravity lies on the symmetry axis (such as shown in Fig. 1a). Tops that have special constraints (e.g., acting under the force of friction) or boundary conditions (e.g., velocity dependent) are called non-holonomic tops and have no complete analytic solutions. In the 20th century, the mathematical investigation of such tops has moved the discourse from the realm of differential equations only to the domains of algebraic geometry and topology [1].

Tops In Art

No doubt many people think of tops as mere children’s playthings. For thousands of years they certainly provided children with engaging activity. Nonetheless, artists as prominent as Peter Brueghel the Elder featured them in their paintings (Fig. 4a). Tops continue to be included in artworks (paintings, prints and sculptures) in both the West and in the Orient. In the 20th century, prominent artists such as Max Ernst employed them to convey less obvious messages (Fig. 4b). And around the world, giant (non functional) top sculptures have even been erected in China, New Zealand, the Netherlands (Fig. 4c) and elsewhere.

Figures 4a (l.), 4b (c.) & 4c (r.): *(a) From the painting “Children’s Games” by Pieter Brueghel, 1560; (b) “Ubu Imperator” by Max Ernst 1923; (c) 10 meter tall top sculpture by Dutch artist Peter Hohmann.*

Tops have also figured in other art forms. Franz Kafka wrote a typically enigmatic parable about them entitled “Der Kreisel”, and Hans Christian Andersen turned the life of a top into a fable with a moral in “The Top and Ball”. Nick Bantock’s 1991 “The History of Imaginary Spinning Tops” is a miniature tour de force that combines art with counterfactual history. Surely the most famous appearance of gyroscopic tops in all of literature is in Lewis Carroll’s beginning lines of “Jabberwocky”: “’Twas brillig, and the slithy toves did gyre and gimble in the wabe.” In music, Georges Bizet wrote a short, nervous piano piece called “La Toupie” that is likely to make your ears spin.

Figures 5a (l.), 5b (c.) & 5c (r.): *(a) Device used in Marcel Duchamp’s film “Anemic Cinema”, 1926; (b) frame from Charles and Ray Eames film “Tops”, 1969; (b) frame from the film “Inception”, 2010.*

Spinning tops have been featured in films as well. In 1926, the polymath (painter, kinetic artist, chess player) Marcel Duchamp devised several kinetic objects that employed spin to create their effects (Fig. 5a). Though not strictly speaking tops, their rotary motion is used for the same effect as several optical tops discussed in a later section. The object shown in Fig. 5a was used to produce the visual effects in Duchamp's (almost palindrome titled) 1926 film "Anemic Cinema". The great 20th century designers Charles and Ray Eames choreographed and shot arguably the finest film ever made about tops (Fig. 5b). Not a word is spoken: the music and dance of the tops speak for themselves. And a (rather poorly designed) top (Fig. 5c) figures prominently in the recent movie "Inception" where it is used to help the protagonist distinguish reality from a dream! And there are at least two museums devoted solely to spinning tops: one is in the U.S. in Wisconsin; the other - with over 20,000 tops - is in Nagoya, Japan.

Tops As Art

What is "Art" (with a capital A)? It is probably fair to say that no one can really assert with authority what "Art" is and what is "only" craft; or what is simply an artifact of the era and what is just an amusement. Shown below are a few examples of tops that certainly display a great deal of artistry. They also exhibit tremendous technical craftsmanship (unlike much "Art" that is produced at the present time).

Figures 6a (l.), 6b (c.) & 6c (r.): *(a) Finger top by American Randy Rhine (9.5 cm tall); (b) finger top by Russian Irina Bobkova (10 cm tall); (c) supported top by American Richard Patterson (60 cm tall).*

Some objects categorized as "kinetic art" certainly share many characteristics with more traditional spinning tops. They can spin or gyrate to produce a combination of motions in response to an external torque. And just like less elaborate children's tops, they engage both the hand and the mind.

Figures 7a (l.), 7b (c.) & 7c (r.): *(a) "Gyroscope" brass sculpture by Israeli Naim Basson (25 cm tall); (b) anonymous kinetic sculpture (35 cm tall); (c) "Chaotic Gyroscope" by Marc Broda (20 cm tall).*

Tops in Mind

In 1730, Isaac Newton published his great book "Optics" in which he presented his experiments using prisms to separate white light into its component colors and then to recombine the colors back into white again. He also devised a color disk (Fig. 8a) with seven colors which, when spun rapidly, would combine the separate color sectors and lead to the appearance of white. He fully realized that color is a visual phenomenon and wrote: "For the Rays to speak properly are not coloured". In 1855, the 24 year-old British physicist James Clerk Maxwell also designed a spinning disk (Fig. 8b), but this time to show that all colors could be produced using red, green and blue only. His quantitative experimental spinning disk studies were one of the cornerstones of the currently accepted trichomaticity theory of color vision.

Figures 8a (l.), 8b (c.) & 8c (r.): *(a) Modern version of Isaac Newton's color mixing wheel; (b) image of James Clerk Maxwell's color mixing top, 1855; (c) Project LITE CD Benham finger top for color, 2012.*

Somewhat earlier - also using a spinning disk but consisting only of black and white striped patterns - British physicist David Brewster noticed the appearance of color. This so-called "flicker" color can most readily be produced by spinning a black and white finger top (Fig. 8c) that was first popularized in 1895 by British toy maker Charles Benham. The great German physicist Ernst Mach also employed a spinning top (Fig. 9a) to display the purely psychophysical appearance of non-existent patterns that are now called "Mach Bands". Other optical tops can produce a variety of visual effects including a strong motion after-effect of counter-rotation upon the cessation of the top's spinning. Artists also have employed tops for combined visual and artistic effects. The German Bauhaus artist Ludwig Hirschfield Mack adapted some of these same color top (Fig. 9c) ideas for use in teaching artists and designers about color. And the tradition of using spinning disks to demonstrate visual phenomena was employed by Marcel Duchamp in his "Rotoreliefs" (Fig. 9b), which can induce in the viewer a strong impression of depth.

Figures 9a (l.), 9b (c.) & 9c (r.): *(c) Project LITE CD finger top for producing Mach bands, 2012; (b) Marcel Duchamp's Rotoreliefs with turntable, 1935; (c) Bauhaus "Optical Color-Mixer", 1924.*

Applied Top(ic)s

It took a little over a century for spinning tops to go from the playroom to the war room. Johann Bohnenberger, a German professor at the University of Tubingen, invented the gyroscope (Fig. 10a) around 1810. It is essentially a top supported at two ends, but able to move about in three dimensions. His publication about it appeared in 1817. Initially it was used as a lecture demonstration apparatus to illustrate mathematical and physical aspects of inertia by using a top. In 1852 the French scientist Jean Bernard Leon Foucault gave the device its name and used it to give a non-astronomical demonstration of the rotation of the Earth. A few years later, James Clerk Maxwell devised what may well be the finest demonstration top ever made (Fig. 10b). Unlike most of the tops shown previously, it is supported at its center of mass and its mass distribution is adjustable. It is exquisitely sensitive to any external torque. But it took the development of new materials, good bearings and the electric motor during the next half-century to allow for a long-spinning top which could be employed for practical purposes. With the rise of steel ships and submarines at the end of the 19th century, a new means of non-magnetic navigation was required. Between roughly 1900 and 1916, a reliable gyrocompass based on the inertial fixed pointing properties of the gyroscope was developed essentially independently in Germany by Hans Anschutz-Kaempfe and in America by Elmer Sperry. This led to a long patent fight between the two inventors that was finally decided with the expert testimony of a formerly obscure Swiss patent clerk: Albert Einstein. A century later, the most expensive top ever made (Fig. 10c) was flown in space to test his General Theory of Relativity which was - ironically - first published in the same year as the patent fight. The Gravity Probe B experiment itself took over half a century to complete at a cost of $750 million. The application of gyroscopes to ship stabilization, as well as to air, ship and guided missile navigation, has grown over the past hundred years. Spinning tops have also pointed the way for space ships heading for other worlds.

Figures 10a (l.), 10b (c.) & 10c (r.): *(a) Bohnenberger Gyroscope ca. 1817 (16 cm tall); (b) James Clerk Maxwell's dynamical top 1858 (20 cm tall); (c) Gravity Probe B Gyroscope ball (~4 cm diameter).*

Counter (Intuitive) Tops

The studies of tops through the end of the 19th century usually ignored frictional forces, as well as the elastic properties of real tops. These play havoc with simple analytic treatments of their motions. A number of tops have been developed that continue to challenge (and defeat!) simple analytic treatment and intuitive understanding. The simplest of these is a spinning coin. Spun about its symmetry axis, it will begin to settle down and start rolling as well. The combination of spin and rolling (which has been dubbed "spolling") leads to a very curious combined motion. As time goes by, the spin rate slows down, and the point of contact between the coin and the ground leads to a buzzing sound with an ever-increasing frequency. Eventually the coin's motion stops. A commercial version of it is sold as "Euler's Disk". Even more counter-intuitive is the motion of a top that was invented and patented at the end of the 19th century. It is called a tippe-top or tippy-top (Fig. 11a). Once spun, it will invert and spin upside down in the opposite direction. No analytic solution for its motion has been completed to date.

Figures 11a (l.), 11b (c.) & 11c (r.): *(a) Wood tippe-top (7 cm diam.); (b) bronze rattleback made by German artist Jack Mankiewicz (26 cm length); (c) traditional Tibetan singing bowl (10 cm diam.).*

The rattleback (also called a celt, anagyre or wobblestone) (Fig. 11b) was also first studied at the end of the 19th century. It has a long ellipsoidal body with either a slightly asymmetric shape or an asymmetric mass distribution. When spun about the central axis perpendicular to its length in, say, a clockwise direction it may continue to spin until stopped by friction. Spun in the opposite direction, it will change from a spinning motion to a rattling motion, and then turn around and spin in the opposite direction. After over a century of analysis, only numerical treatments provide a quantitative description of its motion.

To this list of novel tops, I now add a new kind of top that appears never to have been reported on or discussed in the literature until now. If one takes a bowl with a rounded bottom, like the one shown in Fig. 11c, and spins it horizontally around its point of contact with a surface, a remarkable motion occurs. At first it simply spins, but quite soon it begins to wobble and rise up. This combination of spinning and wobbling I propose to call "spobbling". It is the opposite of "spolling" in the sense that when the effect occurs, the bowl rises instead of falls. How does it work? The analysis is left as an exercise for the reader.

Summary

Tops continue to intrigue people even after three millennia of playing with them. Many mathematical problems concerning their dynamics remain unresolved. There are surely more tops waiting to be devised. Tops provide a nice bridge between art and mathematics. And they entertain both "mens et manus".

Acknowledgments

Some of the ideas in this paper grew out of "Project LITE: Light Inquiry Through Experiments" in which top patterns were designed to be mounted on CD's for spinning as tops. Those patterns can be found at http://lite.bu.edu. Project LITE was supported in part by NSF Grant # DUE - 0715975. I have enjoyed interacting with Randy Rhine, Richard Bischoff, Lourens Bas, Jack Mankiewicz, Don Olney, Guido Sandri and other members of the small community of people who might each be dubbed "topaholics".

References

[1] M. Audin, *Spinning Tops: A course on Integrable Systems*, Cambridge U. Press, Cambridge, 1996.
[2] L. Bas, *The Lost Art of Spinning Tops*, J. A. Verdoorn, Netherlands, 2011.
[3] D. W. Gould, *The Top: Universal Toy, Enduring Pastime*, Clarkson N. Potter, Inc., New York, 1973.
[4] F. Klein & A. Sommerfeld, *The Theory of the Top*, Volumes I - IV, Birkhauser, Boston, 2008 - 2014.