Physics of the PhiTOP®

Kenneth Brecher, Boston University, Boston, MA
Rod Cross, University of Sydney, Sydney, Australia

The PhiTOP® (or ΦTOP®) is a physics toy designed not only to act as a spinning top but also to appeal to the eye and to the scientifically curious mind.[1] It is currently made in two versions, one from solid aluminum and the other solid brass. Each top is highly polished, and is elliptical in one cross section and circular in another. It is therefore a prolate ellipsoid or a spheroid, as indicated in Fig. 1. Its name derives from the ratio of the lengths of the major to minor axes, which is equal to the golden mean $\Phi = (1 + \sqrt{5})/2 \sim 1.618$. It functions in the same way as a spinning egg or a spinning football in that it rises on one end if it is initially set spinning at sufficient speed with its long axis horizontal. The center of mass rises in an unexpected and somewhat mysterious manner, an effect that it also shares with a tippe top.

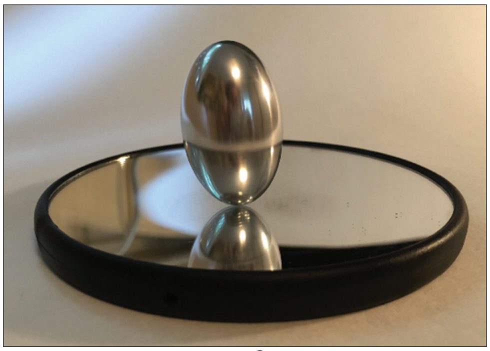

Fig. 1 An upright PhiTOP® spinning on a curved mirror.

There are many articles in the physics literature devoted to spinning eggs and tippe tops, most of which are highly mathematical.[2-4] The general conclusion is that the friction force acting at the bottom of an egg or a tippe top exerts a torque that results in a rise of the center of mass. That explanation is comforting, but it is not very transparent. Consider the situation shown in Fig. 2 where a PhiTOP is spinning about a vertical axis at angular velocity Ω and is spinning about its long axis at angular velocity ω. When the top is first set spinning, Ω is typically much larger than ω, with the result that the contact point P on the horizontal surface slides out of the page. As a result, a horizontal friction force $\boldsymbol{F}$ acts into the page at P.

Suppose that r is the perpendicular distance from P to the long axis, R is the perpendicular distance from P to the vertical axis (Z), and h is the perpendicular distance from P to the X-axis through the center of mass, G. The torque due to the friction force acting about the center of mass has three separate components. The component FR acts to decrease Ω with time. The component Fr acts to increase ω with time. The component Fh must also do something, but therein lies the essence of the problem. Since Fh is directed along the X-axis, intuition suggests that the top might rotate around the X-axis. However, intuition is notoriously misleading when it comes to spinning things.

Consider, for example, the effect of the vertical reaction force $\boldsymbol{N}$ acting in the Z-direction through P. Provided that the top is rising slowly, N is closely equal to Mg. The torque about G due to N is equal to MgR and it acts into the page along the positive Y-axis through G. Intuition might predict that the top should rotate about the Y-axis, and fall onto the horizontal surface instead of rising, as it would in Fig. 2 if the top was not spinning. Intuition is clearly wrong. Instead, the top rotates about the vertical axis like any other spinning top, an effect known as precession.

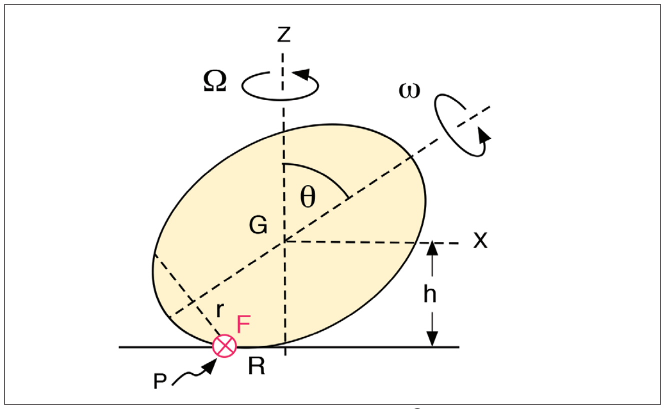

Fig. 2. Geometry of a spinning PhiTOP®. *F* is the friction force into the page.

When the top is first set in motion, it is set spinning rapidly at angular velocity Ω about the vertical axis by an external torque, typically applied by the thumb of one hand and the first finger of the other hand. Friction acts to decrease Ω but it does not rapidly decrease to zero. Instead, Ω decreases to a steady value when the top stops rising, at which point the torque MgR is equal to the rate of change of the X-component of the angular momentum, L_X. The magnitude of L_X remains constant when Ω is constant, but the torque acts to change its direction. That is, $MgR = \Delta L_X/\Delta t = \Omega L_X$, which determines the steady value of the precession frequency. The same relation applies to a spinning top or a gyroscope, as described in most elementary physics textbooks.

Given that the normal reaction force in the Z-direction acts to sustain steady precession about the Z-axis, it would be reasonable to suppose (or guess) that the friction force in the Y-direction might cause the top to precess about the Y-axis. The equations of motion for a spinning egg or a tippe top are quite complicated, but if small terms are neglected, it can be shown[5] that $Fh = -\Delta L_Y/\Delta t = -\Omega L_Y$ to a very good approximation, where $L_Y = I_1\, d\theta/dt$ is the Y-component of the angular momentum and I_1 is the moment of inertia for rotation about the Y-axis. That is, the torque in the X-direction is equal to the negative rate of change of L_Y due to rotation of the top about the Z-axis. The negative sign has a simple geometric interpretation. If a vector in the X-direction is rotated through a small angle about the Z-axis, the change in the vector is in the Y-direction. If a vector in the Y-direction is rotated the same way, the change is in the $-X$-direction. Given that F, h, and Ω

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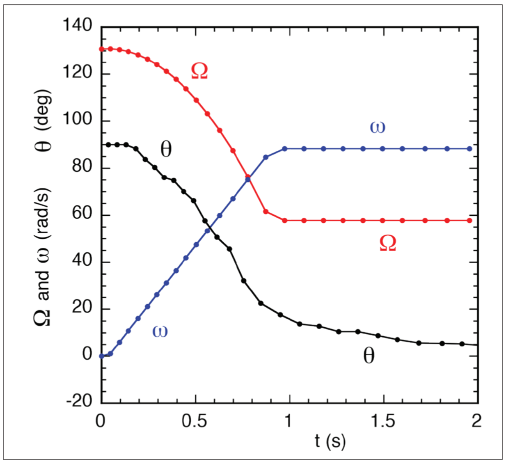

Fig. 3. Results obtained with an aluminum PhiTOP.

are positive quantities, L_Y is negative so θ decreases with time and the top rises.

A typical experimental result with an aluminum PhiTOP is shown in Fig. 3. A more extensive set of experimental results is presented in Ref. 5. The top was set spinning by hand on a smooth table and was filmed at 300 frames/s to measure the rotation speeds and the rate of rise. Marks were drawn on the top to determine the rotation angles and the video film was analyzed with Tracker software. For this top, M = 70.8 g, the major diameter is 50.8 mm, the minor diameter is 31.4 mm, and $I_1 = 1.26\times10^{-5}$ kg·m^2. The value of h therefore increased from about 16 mm to about 25 mm as the top was rising. Taking an average value of h of 20 mm while the top rises and a coefficient of sliding friction $\mu = 0.15$ gives Fh = 0.002 Nm. Since the average value of Ω was about 100 rad/s, the expected rate of rise is $d\theta/dt$ = 1.6 rad/s. The actual rate of rise was about 1.5 rad/s, consistent with the assumed value of μ and the value of μ measured when aluminum slides on plastic.[6]

The experimental result is therefore consistent with the simple explanation that the top rises as a result of precession about the Y-axis, for the same basic reason that the top precesses about the Z-axis. The rapid rise of the top ceases and the rotation frequencies Ω and ω settle into constant values when the top stops sliding and starts rolling on the table. The rolling condition is given by $R\Omega = r\omega$ since the contact point then comes to rest on the surface. If the friction force were to drop to zero, precession about the Y-axis would stop, $d\theta/dt$ would be zero, and there would be no further change in Ω or ω. If the top is spun at low frequency, rolling commences sooner and the top fails to rise as high as it does with a high frequency spin.

Many additional questions can be asked about the top, and could be answered as student projects or found in Ref. 5. For example, is angular momentum and total energy conserved, even approximately, as the top rises, or does the friction force interfere significantly with the conservation laws? How does the final spin compare with the initial spin? Does the rise time increase as the spin increases, as predicted? Is the coefficient of sliding friction, as deduced from the rise time, consistent with the rate of decrease of Ω and the rate of increase of ω? What differences might arise if a brass top is used rather than an aluminum top? And why does θ decrease with time with a spheroid but it increases with time with a tippe top? The short answer to the last question is that the friction force is in the opposite direction.[7] A gyroscope also rises when a horizontal force is applied, as shown in the supplementary video.[8]

References

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Rod Cross *is an honorary (retired) staff member at Sydney University, with interests in the physics of sport and physics education.*
rodney.cross@sydney.edu.au

Kenneth Brecher *received his BS and PHD degrees, both in physics, from M.I.T., where he was also on the faculty from 1972–1979. He was a professor of astronomy and physics from 1979–2017 at Boston University, where he is currently Emeritus Professor.*